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Transient motion of an anisotropic elastic half-space due to a buried line source

Kuang-Chong Wu *, Shyh-Haur Chen

Institute of Applied Mechanics, National Taiwan University, Taipei 106, Taiwan, ROC

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Abstract

A method to deal with the two-dimensional transient problem of a line force or dislocation in an anisotropic elastic half-space is developed. The proposed formulation is similar to Stroh's formalism for anisotropic elastostatics in that the two-dimensional anisotropic elastodynamic problem is cast into a six-dimensional eigenvalue problem and the solution is expressed in terms of the eigenvalues and eigenvectors. An analytic solution is obtained without performing integral transforms. Numerical examples are presented for a silicon half-space subjected to a line force or dislocation.

Keywords: Transient motion; Anisotropic elastic material; Half-space; Buried source

1. Introduction

The propagation and reflection of waves in an elastic half-space is of practical importance in the fields of seismology and non-destructive testing. Lamb (1904) was the first to consider the generation of elastic waves by the application of a surface impulsive line or point force on the surface of an isotropic half-space. He also gave the formal solutions for a buried force as integrals which were later studied by Nakano (1925) and Lapwood (1949), among others.

The two-dimensional Lamb's problem for a transversely isotropic half-space subjected to a surface line force has been studied by Kraut (1963) using Cagniard's technique. The treatment has been extended to general anisotropic materials by Burridge (1971). Payton (1983) has obtained explicit closed form solution for the surface displacements for transversely isotropic media. The interior response was calculated for a

* Corresponding author. Tel.: +888233665695; fax: +888233665696.
E-mail address: wukc@spring.iam.ntu.edu.tw (K.-C. Wu).

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half-space of cubic symmetry by Mourad et al. (1996). Maznev and Every (1997) employed the Fourier transform to show a functional equivalence for surface response between the time and Fourier domain. Recently, Wu (2000) has used a Stroh-like formulation that does not require integral transform to derive explicit solution for the displacement fields.

All of the aforementioned works on anisotropic half-spaces are for surface loading. There appear to be few results for internal sources. Payton (1983) obtained a closed form expression for the epicenter displacement due to a buried point force in a transversely isotropic half-space. Spies (1997) gave the solution in the Fourier transform domain for a point force in a general anisotropic half-space. Recently Wu (2001) derived the surface motion due to a line force or dislocation in a general anisotropic elastic half-space.

In this paper an explicit solution is provided for the interior response due to a impulsive line force or a line dislocation in a general anisotropic half-space. The problem of a buried force is more complicated than that of a surface force as the former involves a characteristic length—the depth of the source. A formulation developed by Wu (2000) will be extended to treat the present problem. In this formulation the solution is expressed in terms of the eigenvalues and eigenvectors of a six-dimensional matrix, which is a function of the material constants, time and position. A major advantage of the proposed formulation is that no integral transforms are required. This fact greatly facilitates derivations of explicit solutions.

The plan of the paper is as follows. In Section 2 the formulation is developed to treat the non-self-similar problems. In Section 3 the formulation is applied to study the problem of a buried source in a traction-free half-space. Numerical examples are given in Section 4. Some concluding remarks are finally given.

2. Formulation

For two-dimensional deformation in which the Cartesian components of the stress σ_{ij} and the displacement u_i, $i,j = 1,2,3$, are independent of x_3, the equations of motion are

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = \rho\ddot{\mathbf{u}}, \tag{1}$$

where $\mathbf{t}_1 = (\sigma_{11}, \sigma_{21}, \sigma_{31})^{\mathrm{T}}$, $\mathbf{t}_2 = (\sigma_{12}, \sigma_{22}, \sigma_{32})^{\mathrm{T}}$, $\ddot{\mathbf{u}}$ is the acceleration, ρ is the density, a subscript comma denotes partial differentiation with respect to coordinates and overhead dot designates derivative with respect to time t. The stress–strain laws are

$$\mathbf{t}_1 = \mathbf{Q}\mathbf{u}_{,1} + \mathbf{S}\mathbf{u}_{,2}, \tag{2}$$

$$\mathbf{t}_2 = \mathbf{S}^{\mathrm{T}}\mathbf{u}_{,1} + \mathbf{T}\mathbf{u}_{,2}, \tag{3}$$

where the matrices **Q, S**, and **T** are related to elastic constants C_{ijks} by

$$Q_{ik} = C_{i1k1}, \quad S_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}.$$

The equations of motion expressed in terms of displacements are obtained by substituting Eqs. (2) and (3) into Eq. (1) as

$$\mathbf{Q}\mathbf{u}_{,11} + (\mathbf{S} + \mathbf{S}^{\mathrm{T}})\mathbf{u}_{,12} + \mathbf{T}\mathbf{u}_{,22} = \rho\ddot{\mathbf{u}}. \tag{4}$$

Let the displacement be assumed as $\mathbf{u}(x_1, x_2, t) = \mathbf{u}(w)$ with the variable $w(x_1, x_2, t)$ implicitly defined by

$$wt - x_1 - p(w)x_2 - q(w) = 0, \tag{5}$$

where $p(w)$ and $q(w)$ are functions of w. It will be shown later that $p(w)$ is determined by Eq. (4) but $q(w)$ is arbitrary. The special case $q(w) = 0$ has been discussed by Wu (2000). The first derivatives of $\mathbf{u}(w)$ with respect to x_1, x_2, and t can be expressed as

$$\mathbf{u}_{,1} = \frac{\partial w}{\partial x_1}\mathbf{u}'(w), \quad \mathbf{u}_{,2} = p(w)\frac{\partial w}{\partial x_1}\mathbf{u}'(w), \quad \dot{\mathbf{u}} = -w\frac{\partial w}{\partial x_1}\mathbf{u}'(w) \tag{6}$$

and the second derivatives as

$$\mathbf{u}_{,11} = \frac{\partial w}{\partial x_1}\frac{\partial}{\partial w}\left(\frac{\partial w}{\partial x_1}\mathbf{u}'(w)\right), \quad \mathbf{u}_{,22} = \frac{\partial w}{\partial x_1}\frac{\partial}{\partial w}\left(p(w)^2\frac{\partial w}{\partial x_1}\mathbf{u}'(w)\right), \tag{7}$$

$$\mathbf{u}_{,12} = \frac{\partial w}{\partial x_1}\frac{\partial}{\partial w}\left(p(w)\frac{\partial w}{\partial x_1}\mathbf{u}'(w)\right), \quad \ddot{\mathbf{u}} = \frac{\partial w}{\partial x_1}\frac{\partial}{\partial w}\left(w^2\frac{\partial w}{\partial x_1}\mathbf{u}'(w)\right), \tag{8}$$

where $\frac{\partial w}{\partial x_1}$ is given by

$$\frac{\partial w}{\partial x_1} = \frac{1}{t - p'(w)x_2 - q'(w)} \tag{9}$$

and 'prime' denotes the derivative with respect to w. With Eqs. (7) and (8), Eq. (4) becomes

$$\frac{\partial w}{\partial x_1}\frac{\partial}{\partial w}\left\{[\mathbf{Q} - \rho w^2\mathbf{I} + p(w)(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) + p(w)^2\mathbf{T}]\frac{\partial w}{\partial x_1}\mathbf{u}'(w)\right\} = \mathbf{0}, \tag{10}$$

where $\mathbf{I}$ is the identity matrix. Let $\mathbf{u}'(w)$ be expressed as

$$\mathbf{u}'(w) = f(w)\mathbf{a}(w), \tag{11}$$

where $f(w)$ is an arbitrary scalar function of w. It follows that $\mathbf{u}(w)$ is a solution of Eq. (4) if

$$\mathbf{D}(p, w)\mathbf{a}(w) = \mathbf{0}, \tag{12}$$

where $\mathbf{D}(p, w)$ is given by

$$\mathbf{D}(p, w) = \mathbf{Q} + p(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) + p^2\mathbf{T} - \rho w^2\mathbf{I}. \tag{13}$$

For non-trivial solutions of $\mathbf{a}(w)$ we must have

$$|\mathbf{D}(p, w)| = 0, \tag{14}$$

where $|\mathbf{D}|$ is the determinant of $\mathbf{D}$.

Eq. (14) provides six eigenvalues of p as a function of w, denoted by $p_k(w)$, $k = 1, 2, \ldots, 6$. The function $p_k(w)$ is single-valued if w is allowed to range over the six sheets $\sum^k$ of its Riemann surface, taking the values $p_k(w)$ on $\sum^k$ (Willis, 1973). If w is real and $|w|$ is sufficiently large, there are six real roots $p_k(w)$ such that (Wu, 2000)

$$\frac{\mathrm{d}w}{\mathrm{d}p} = r_2, \tag{15}$$

where r_2 is the x_2 component of the ray velocity. Three of these roots characterized by $\mathrm{d}w/\mathrm{d}p > 0$ are associated with the rays propagating in the direction of positive x_2 direction and the others by $\mathrm{d}w/\mathrm{d}p < 0$ with the rays propagating in the direction of negative x_2 direction. The three of the former type will be assigned to the Riemann surfaces $\sum^k (k = 1, 2, 3)$ and the three of the latter type to $\sum^k (k = 4, 5, 6)$. The sheets are connected across appropriate lines joining the branch points of $p_k(w)$, which are located on the real axis in the complex w-plane and are determined by $\mathrm{d}w/\mathrm{d}p = 0$. It can be shown that $p_k(w)$ has positive imaginary part in the upper half of $\sum^k (k = 1, 2, 3)$ and negative imaginary part in the upper plane of $\sum^k (k = 4, 5, 6)$. The variable $w_k = w_k(x_1, x_2, t)$ can then be solved from Eq. (5) by taking $p(w) = p_k(w)$.

It is obvious that if $p_k(w)$ and $\mathbf{a}_k(w)$ are, respectively, the eigenvalue and eigenvector of Eq. (12), so are $\overline{p_k(w)}$ and $\overline{\mathbf{a}_k(w)}$, where the superposed bar denotes the complex conjugate. Thus from Eq. (6), the general solution of Eq. (4) may be represented as

$$\mathbf{u}(x_1,x_2,t)_{,1} = 2\mathrm{Re}\left\{\sum_k f_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{a}_k(w_k)\right\}, \tag{16}$$

$$\mathbf{u}(x_1,x_2,t)_{,2} = 2\mathrm{Re}\left\{\sum_k P_k(w_k)f_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{a}_k(w_k)\right\}, \tag{17}$$

$$\dot{\mathbf{u}}(x_1,x_2,t) = -2\mathrm{Re}\left\{\sum_k w_k f_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{a}_k(w_k)\right\}, \tag{18}$$

where $f_k(w_k)$ is an arbitrary function of w_k and $k = 1,2,3$ or $4,5,6$. The choice of the range of k depends on whether up-going rays or down-going rays are considered.

By substituting Eqs. (16) and (17) into Eqs. (2) and (3), the general solutions of the stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed as

$$\mathbf{t}_1(x_1,x_2,t) = 2\mathrm{Re}\left\{\sum_k f_k(w_k)\left[\rho w_k^2\frac{\partial w_k}{\partial x_1}\mathbf{a}_k(w_k) - p_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{b}_k(w_k)\right]\right\}, \tag{19}$$

$$\mathbf{t}_2(x_1,x_2,t) = 2\mathrm{Re}\left\{\sum_k f_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{b}_k(w_k)\right\}, \tag{20}$$

where

$$\mathbf{b}_k(w) = (\mathbf{S}^{\mathrm{T}} + p_k(w)\mathbf{T})\mathbf{a}_k(w) = -\frac{1}{p}(\mathbf{Q} - \rho w^2\mathbf{I} + p_k(w)\mathbf{S})\mathbf{a}_k(w). \tag{21}$$

The second identity in Eq. (21) follows from Eq. (12). Eq. (21) can be cast into the following six-dimensional eigenvalue problem

$$\mathbf{N}(w)\boldsymbol{\xi}(w) = p(w)\boldsymbol{\xi}(w), \tag{22}$$

where

$$\mathbf{N}(w) = \begin{pmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3(w) & \mathbf{N}_1^{\mathrm{T}}\end{pmatrix}, \quad \boldsymbol{\xi}(w) = \begin{pmatrix}\mathbf{a}(w)\\ \mathbf{b}(w)\end{pmatrix},$$
$$\mathbf{N}_1 = -\mathbf{T}^{-1}\mathbf{S}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1}, \quad \mathbf{N}_3(w) = \mathbf{S}\mathbf{T}^{-1}\mathbf{S}^{\mathrm{T}} - \mathbf{Q} + \rho w^2\mathbf{I}.$$

Eq. (22) is in the same form as that in Stroh's formalism for steady-state motion (Stroh, 1962). Let

$$\mathbf{A}(w) = [\mathbf{a}_1(w) \quad \mathbf{a}_2(w) \quad \mathbf{a}_3(w)], \quad \mathbf{B}(w) = [\mathbf{b}_1(w) \quad \mathbf{b}_2(w) \quad \mathbf{b}_3(w)], \tag{23}$$

$$\widehat{\mathbf{A}}(w) = [\mathbf{a}_4(w) \quad \mathbf{a}_5(w) \quad \mathbf{a}_6(w)], \quad \widehat{\mathbf{B}}(w) = [\mathbf{b}_4(w) \quad \mathbf{b}_5(w) \quad \mathbf{b}_6(w)]. \tag{24}$$

These matrices satisfy the closure relations (Ting, 1996, p. 445)

$$\mathbf{A}(w)\mathbf{A}^{\mathrm{T}}(w) + \widehat{\mathbf{A}}(w)\widehat{\mathbf{A}}^{\mathrm{T}}(w) = \mathbf{0} = \mathbf{B}(w)\mathbf{B}^{\mathrm{T}}(w) + \widehat{\mathbf{B}}(w)\widehat{\mathbf{B}}^{\mathrm{T}}(w), \tag{25}$$

$$\mathbf{A}(w)\mathbf{B}^{\mathrm{T}}(w) + \widehat{\mathbf{A}}(w)\widehat{\mathbf{B}}^{\mathrm{T}}(w) = \mathbf{I} = \mathbf{B}(w)\mathbf{A}^{\mathrm{T}}(w) + \widehat{\mathbf{B}}(w)\widehat{\mathbf{A}}^{\mathrm{T}}(w), \tag{26}$$

if the eigenvectors $\boldsymbol{\xi}_\alpha(w)$, $\alpha = 1,2,\ldots,6$, are normalized such that

$$\mathbf{a}_k^{\mathrm{T}}(w)\mathbf{b}_j(w) + \mathbf{b}_k^{\mathrm{T}}(w)\mathbf{a}_j(w) = \delta_{kj},$$

where δ_{kj} is the Kronecker delta

3. A line force and dislocation in a half-space

Consider a line force $\mathbf{F}H(t)$ and a dislocation with Burgers vector $\boldsymbol{\beta}H(t)$, $H(t)$ being the unit step function, at $x_1 = 0$ and $x_2 = h$ in the half-space $x_2 \geqslant 0$. The configuration is shown in Fig. 1. The boundary conditions at $x_2 = 0$ is given by

$$\mathbf{t}_2(x_1, t) = \mathbf{0}. \tag{27}$$

The resulting $\dot{\mathbf{u}}$ and $\mathbf{t}_2$ may be expressed as

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}^{(0)} + \dot{\mathbf{u}}^{(1)}, \tag{28}$$

$$\mathbf{t}_2 = \mathbf{t}_2^{(0)} + \mathbf{t}_2^{(1)}, \tag{29}$$

where $\dot{\mathbf{u}}^{(0)}$ and $\mathbf{t}_2^{(0)}$ are, respectively, the particle velocity and the stress vector due to the sources in an infinite medium, $\dot{\mathbf{u}}^{(1)}$ and $\mathbf{t}_2^{(1)}$ are those due to the reflected waves from the free surface.

The solution for the line force in an infinite medium has been obtained by Wu (2000) and that for the line dislocation may be derived similarly. The result is

$$\dot{\mathbf{u}}^{(0)} = \frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6} c_k(w_k)\frac{\partial w_k}{\partial x_1}\mathbf{a}_k(w_k)\right\}, \tag{30}$$

$$\mathbf{t}_2^{(0)} = -\frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6} \frac{c_k(w_k)}{w_k}\frac{\partial w_k}{\partial x_1}\mathbf{b}_k(w_k)\right\}, \tag{31}$$

where $c_k(w_k) = \mathbf{a}_k^{\mathrm{T}}(w_k)\mathbf{F} + \mathbf{b}_k^{\mathrm{T}}(w_k)\boldsymbol{\beta}$ and w_k is determined by

$$wt = x_1 + p_k(w)(x_2 - h). \tag{32}$$

Eqs. (30) and (31) may also be expressed in matrix form as

$$\dot{\mathbf{u}}^{(0)} = \frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6} \frac{\partial w_k}{\partial x_1}\widehat{\mathbf{A}}(w_k)I_{k-3}\left(\widehat{\mathbf{A}}(w_k)^{\mathrm{T}}\mathbf{F} + \widehat{\mathbf{B}}(w_k)^{\mathrm{T}}\boldsymbol{\beta}\right)\right\}, \tag{33}$$

$$\mathbf{t}_2^{(0)} = -\frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6} \frac{1}{w_k}\frac{\partial w_k}{\partial x_1}\widehat{\mathbf{B}}(w_k)\mathbf{I}_{k-3}\left(\widehat{\mathbf{A}}(w_k)^{\mathrm{T}}\mathbf{F} + \widehat{\mathbf{B}}(w_k)^{\mathrm{T}}\boldsymbol{\beta}\right)\right\}, \tag{34}$$

where $\mathbf{I}_k = \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}$ and $\mathbf{e}_k$ is the unit vector in the x_k-direction.

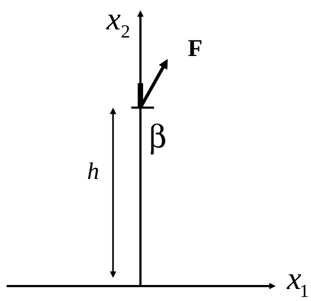

Fig. 1. Configuration of the problem of interest.

From Eqs. (18) and (20), let $\dot{\mathbf{u}}^{(1)}$ and $\mathbf{t}_2^{(1)}$ be expressed as

$$\dot{\mathbf{u}}^{(1)}(x_1,x_2,t) = \frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{c_k(w_{kj})R_{kj}(w_{kj})\frac{\partial w_{kj}}{\partial x_1}\mathbf{a}_j(w_{kj})\right\}, \tag{35}$$

$$\mathbf{t}_2^{(1)}(x_1,x_2,t) = -\frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{\frac{c_k(w_{kj})}{w_{kj}}R_{kj}(w_{kj})\frac{\partial w_{kj}}{\partial x_1}\mathbf{b}_j(w_{kj})\right\}, \tag{36}$$

where w_{kj} is determined by

$$wt = x_1 + p_j(w)x_2 - p_k(w)h \tag{37}$$

and R_{kj} is the reflection coefficient. Note that at $x_2 = 0$,

$$w_{kj}(x_1,t) = w_k(x_1,t) \tag{38}$$

and Eq. (36) becomes

$$\mathbf{t}_2^{(1)}(x_1,t) = -\frac{1}{\pi}\sum_{k=4}^{6}\mathrm{Im}\left\{\frac{c_k(w_k)}{w_k}\frac{\partial w_k}{\partial x_1}\mathbf{B}(w_k)\mathbf{R}_k(w_k)\right\}, \tag{39}$$

where $\mathbf{R}_k(w_k) = [R_{k1}(w_k) \quad R_{k2}(w_k) \quad R_{k3}(w_k)]^{\mathrm{T}}$. Substituting Eqs. (31) and (39) into Eq. (27) yields

$$\mathbf{R}_k(w_k) = -\mathbf{B}^{-1}(w_k)\mathbf{b}_k(w_k). \tag{40}$$

The function $R_{kj}(w_{kj})$ can be obtained from $\mathbf{R}_k$ as

$$R_{kj}(w_{kj}) = \mathbf{e}_j^{\mathrm{T}}\mathbf{R}_k(w_{kj}) = \mathbf{e}_j^{\mathrm{T}}\mathbf{R}(w_{kj})\mathbf{e}_{k-3}, \tag{41}$$

where

$$\mathbf{R}(w) = -\mathbf{B}(w)^{-1}\widehat{\mathbf{B}}(w). \tag{42}$$

With Eq. (41) substituted, Eqs. (35) and (36) can be expressed in matrix form as

$$\dot{\mathbf{u}}^{(1)} = \frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{\frac{\partial w}{\partial x_1}\mathbf{A}(w)\mathbf{I}_j\mathbf{R}(w)\mathbf{I}_{k-3}(\widehat{\mathbf{A}}^{\mathrm{T}}(w)\mathbf{F} + \widehat{\mathbf{B}}^{\mathrm{T}}(w)\boldsymbol{\beta})\right\}_{w=w_{kj}}, \tag{43}$$

$$\mathbf{t}_2^{(1)} = -\frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{\frac{1}{w}\frac{\partial w}{\partial x_1}\mathbf{B}(w)\mathbf{I}_j\mathbf{R}(w)\mathbf{I}_{k-3}(\widehat{\mathbf{A}}^{\mathrm{T}}(w)\mathbf{F} + \widehat{\mathbf{B}}^{\mathrm{T}}(w)\boldsymbol{\beta})\right\}_{w=w_{kj}}. \tag{44}$$

Three special cases: (a) $t \to \infty$, (b) $h \to 0$, and (c) $x_2 = 0$, are discussed as follows:

(a) For $t \to \infty$, $w_{kj} \to 0$ such that

$$\frac{\partial w_{kj}}{\partial x_1} \to \frac{1}{t}, \quad w_{kj}t \to z_j - p_k(0)h$$

where $z_j = x_1 + p_j(0)x_2$. Eqs. (34) and (44) become

$$\mathbf{t}_2^{(0)}(x_1,x_2,t) = \frac{1}{\pi}\mathrm{Im}\left\{\mathbf{B}(0)\left\langle\frac{1}{z_* - p_*(0)h}\right\rangle(\mathbf{A}^{\mathrm{T}}(0)\mathbf{F} + \mathbf{B}^{\mathrm{T}}(0)\boldsymbol{\beta})\right\}, \tag{45}$$

$$\mathbf{t}_2^{(1)}(x_1,x_2,t) = \frac{1}{\pi}\sum_{k=1}^{3}\mathrm{Im}\left\{\mathbf{B}(0)\left\langle\frac{1}{z_* - \bar{p}_k(0)h}\right\rangle\mathbf{B}^{-1}(0)\overline{\mathbf{B}}(0)\mathbf{I}_k(\bar{\mathbf{A}}^{\mathrm{T}}(0)\mathbf{F} + \overline{\mathbf{B}}^{\mathrm{T}}(0)\boldsymbol{\beta})\right\}, \tag{46}$$

where $\left\langle \frac{1}{z_* - p_k(0)h} \right\rangle$ is the diagonal matrix given by

$$\left\langle \frac{1}{z_* - p_k(0)h} \right\rangle = \mathrm{diag}\left(\frac{1}{z_1 - p_k(0)h}, \frac{1}{z_2 - p_k(0)h}, \frac{1}{z_3 - p_k(0)h} \right).$$

In Eqs. (45) and (46), the following replacements have been made:

$$p_{k+3}(0) = \bar{p}_k(0), \quad k = 1,2,3, \qquad \widehat{\mathbf{A}}(0) = \overline{\mathbf{A}}(0), \qquad \widehat{\mathbf{B}}(0) = \overline{\mathbf{B}}(0).$$

Eqs. (45) and (46) agree with the static result (Ting, 1996, pp: 261–262).

(b) When $h \to 0$, $w_{kj} = w_j$, and $\frac{\partial w_{kj}}{\partial x_1} = \frac{\partial w_j}{\partial x_1}$ where w_j and $\frac{\partial w_j}{\partial x_1}$ are given by

$$w_j t = x_1 + p_j(w_j)x_2, \quad \frac{\partial w_j}{\partial x_1} = \frac{1}{t - p_j'(w_j)x_2}.$$

Eq. (43) for $\boldsymbol{\beta} = 0$ can be written as

$$\dot{\mathbf{u}}^{(1)}(x_1, x_2, t) = -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{j=1}^{3} \frac{\partial w_j}{\partial x_1} \mathbf{A}(w_j)\mathbf{I}_j\mathbf{B}^{-1}(w_j)\widehat{\mathbf{B}}(w_j)\widehat{\mathbf{A}}^{\mathrm{T}}(w_j)\mathbf{F} \right\}. \tag{47}$$

On the other hand, Eq. (33) with $h \to 0$ may be expressed as

$$\dot{\mathbf{u}}^{(0)}(x_1, x_2, t) = -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{j=1}^{3} \frac{\partial w_j}{\partial x_1} \mathbf{A}(w_j)\mathbf{I}_j\mathbf{B}^{-1}(w_j)\mathbf{B}(w_j)\mathbf{A}^{\mathrm{T}}(w_j)\mathbf{F} \right\}. \tag{48}$$

The total velocity obtained by adding Eqs. (47) and (48) is given by

$$\dot{\mathbf{u}}(x_1, x_2, t) = -\frac{1}{\pi}\mathrm{Im}\left\{ \sum_{j=1}^{3} \frac{\partial w_j}{\partial x_1} \mathbf{A}(w_j)\mathbf{I}_j\mathbf{B}^{-1}(w_j)\mathbf{F} \right\}, \tag{49}$$

where Eq. (26) has been used. Eq. (49) is identical with the result derived by Wu (2000) for a line force applied on the surface of a half-space.

(c) At $x_2 = 0$, the expression given by Eq. (43) may be simplified as

$$\dot{\mathbf{u}}^{(1)}(x_1, t) = -\frac{1}{\pi}\sum_{k=4}^{6}\mathrm{Im}\left\{ \frac{\partial w_k}{\partial x_1} \mathbf{A}(w_k)\mathbf{B}^{-1}(w_k)\widehat{\mathbf{B}}(w_k)\mathbf{I}_{k-3}(\widehat{\mathbf{A}}^{\mathrm{T}}(w_k)\mathbf{F} + \widehat{\mathbf{B}}^{\mathrm{T}}(w_k)\boldsymbol{\beta}) \right\}. \tag{50}$$

The total surface response obtained by the sum of Eqs. (33) and (50) is

$$\dot{\mathbf{u}}(x_1, t) = \frac{1}{\pi}\sum_{k=4}^{6}\mathrm{Im}\left\{ \frac{\partial w_k}{\partial x_1} (\hat{\mathbf{B}}^{-1}(w_k))^{\mathrm{T}}\mathbf{I}_{k-3}(\hat{\mathbf{A}}^{\mathrm{T}}(w_k)\mathbf{F} + \widehat{\mathbf{B}}^{\mathrm{T}}(w_k)\boldsymbol{\beta}) \right\}. \tag{51}$$

The steps leading to Eq. (51) are given as follows:

$$\begin{aligned}
\widehat{\mathbf{A}}(w_k) - \mathbf{A}(w_k)\mathbf{B}^{-1}(w_k)\widehat{\mathbf{B}}(w_k) &= \left[\widehat{\mathbf{A}}(w_k)\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} - \mathbf{A}(w_k)\mathbf{B}^{-1}(w_k)\widehat{\mathbf{B}}(w_k)\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} \right] \left(\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} \right)^{-1} \\
&= \left[\widehat{\mathbf{A}}(w_k)\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} + \mathbf{A}(w_k)\mathbf{B}^{-1}(w_k)\mathbf{B}(w_k)\mathbf{B}(w_k)^{\mathrm{T}} \right] \left(\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} \right)^{-1} \\
&= [\widehat{\mathbf{A}}(w_k)\widehat{\mathbf{B}}(w_k)^{\mathrm{T}} + \mathbf{A}(w_k)\mathbf{B}(w_k)^{\mathrm{T}}](\widehat{\mathbf{B}}(w_k)^{\mathrm{T}})^{-1} = (\widehat{\mathbf{B}}(w_k)^{\mathrm{T}})^{-1},
\end{aligned}$$

where the second line follows from Eq. (25) and the last line is a result of Eq. (26). Eq. (51) has been obtained by Wu (2001) by a different approach.

In summary, the particle velocity due to the line force and the dislocation given by substituting Eqs. (33) and (43) into Eq. (28) is

$$\dot{\mathbf{u}}(x_1, x_2, t) = \mathbf{G}_{\mathrm{f}}(x_1, x_2, t)\mathbf{F} + \mathbf{G}_{\mathrm{b}}(x_1, x_2, t)\boldsymbol{\beta}, \tag{52}$$

where

$$\mathbf{G}_{\mathrm{f}} = \mathbf{G}_{\mathrm{f}}^{(0)} + \mathbf{G}_{\mathrm{f}}^{(1)}, \tag{53}$$

with $\mathbf{G}_{\mathrm{f}}^{(0)}$ and $\mathbf{G}_{\mathrm{f}}^{(1)}$ given by

$$\mathbf{G}_{\mathrm{f}}^{(0)} = \frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6}\frac{\partial w_k}{\partial x_1}\widehat{\mathbf{A}}(w_k)\mathbf{I}_{k-3}\widehat{\mathbf{A}}(w_k)^{\mathrm{T}}\right\}, \tag{54}$$

$$\mathbf{G}_{\mathrm{f}}^{(1)} = \frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{\frac{\partial w_{kj}}{\partial x_1}\mathbf{A}(w_{kj})\mathbf{I}_j\mathbf{R}(w_{kj})\mathbf{I}_{k-3}\widehat{\mathbf{A}}^{\mathrm{T}}(w_{kj})\right\}, \tag{55}$$

and

$$\mathbf{G}_{\mathrm{b}} = \mathbf{G}_{\mathrm{b}}^{(0)} + \mathbf{G}_{\mathrm{b}}^{(1)}, \tag{56}$$

with $\mathbf{G}_{\mathrm{b}}^{(0)}$ and $\mathbf{G}_{\mathrm{b}}^{(1)}$ given by

$$\mathbf{G}_{\mathrm{b}}^{(0)} = \frac{1}{\pi}\mathrm{Im}\left\{\sum_{k=4}^{6}\frac{\partial w_k}{\partial x_1}\widehat{\mathbf{A}}(w_k)\mathbf{I}_{k-3}\widehat{\mathbf{B}}(w_k)^{\mathrm{T}}\right\}, \tag{57}$$

$$\mathbf{G}_{\mathrm{b}}^{(1)} = \frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\left\{\frac{\partial w_{kj}}{\partial x_1}\mathbf{A}(w_{kj})\mathbf{I}_j\mathbf{R}(w_{kj})\mathbf{I}_{k-3}\widehat{\mathbf{B}}^{\mathrm{T}}(w_{kj})\right\}. \tag{58}$$

Since the velocity field due to $\mathbf{F}H(t)$ is the same as the displacement field due to $\mathbf{F}\delta(t), \delta(\,)$ being the Dirac delta function, $\mathbf{G}_{\mathrm{f}}$ may also be regarded as the Green's function for an impulsive force. Similarly $\mathbf{G}_{\mathrm{b}}$ is the Green's function due to an impulsive dislocation.

4. Numerical examples

In order to calculate the dynamic response due to a buried source, w_k and w_{kj} as a function of x_1, x_2 and t must be determined from Eqs. (32) and (37), respectively. Since those functions cannot be obtained explicitly in general, a numerical scheme is developed as follows. Either Eq. (32) or Eq. (37) is in the following form:

$$\phi(w) = wt - x_1 - p(w)x_2 - q(w) = 0 \tag{59}$$

For fixed x_1, x_2 and t, expand $\phi(w)$ about some trial value w_0 up to the second order term by Taylor's series,

$$\phi(w) \approx \phi_0 + \left(\frac{\partial\phi}{\partial w}\right)_0 \Delta w + \frac{1}{2}\left(\frac{\partial^2\phi}{\partial w^2}\right)_0 (\Delta w)^2 \tag{60}$$

where $\Delta w = w - w_0$, and $(f)_0 = f(w_0)$. An approximate solution of Δw can be obtained by substituting Eq. (60) into Eq. (59) as

$$\Delta w = (-b + \sqrt{b^2 - ac})/a \tag{61}$$

where a, b, and c are given by

$$a = -p''(w_0)x_2 - q''(w_0),\ b = t - p'(w_0)x_2 - q'(w_0),\ c = 2\phi_0.$$

Let $w_1 = w_0 + \Delta w$. If $|\phi(w_1)| < \epsilon$, where ϵ is a preset error, then w_1 is accepted as the solution of w for given x_1, x_2 and t. Otherwise Eq. (61) is used again with w_0 replaced by w_1. The process is repeated until the error criterion is met.

In the numerical examples the Green's functions are expressed in the following dimensionless form:

$$\overline{\mathbf{G}}_\mathrm{f}(n,m,\tau) = \pi\rho c_0 h \mathbf{G}_\mathrm{f}(x_1,x_2,t), \quad \overline{\mathbf{G}}_\mathrm{b}(n,m,\tau) = \frac{\pi h}{c_0}\mathbf{G}_\mathrm{b}(x_1,x_2,t) \tag{62}$$

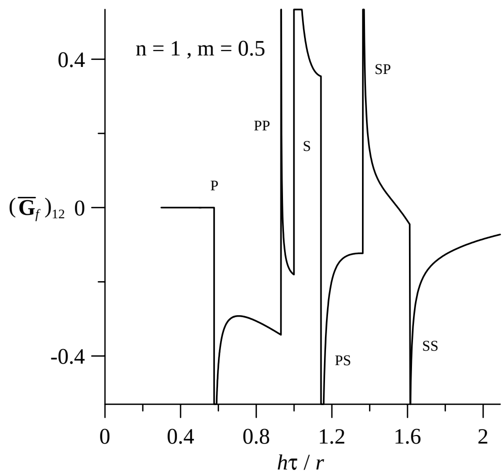

Fig. 2. $(\overline{\mathbf{G}}_\mathrm{f})_{12}$ for the isotropic material at $n = 1$ and $m = 0.5$.

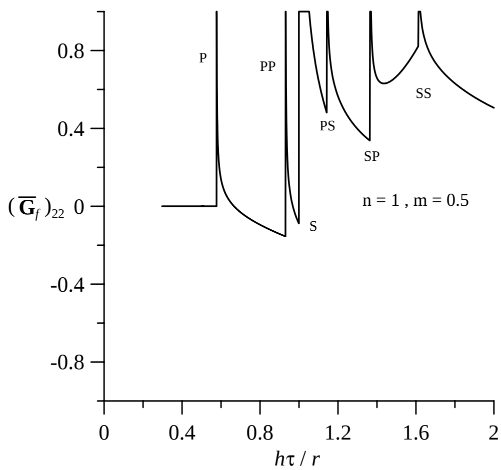

Fig. 3. $(\overline{\mathbf{G}}_\mathrm{f})_{22}$ for the isotropic material at $n = 1$ and $m = 0.5$.

where $n = x_1/h, m = x_2/h, \tau = tc_0/h, c_0 = \sqrt{C_{44}/\rho}$. Some components of $\overline{\mathbf{G}}_f$ and $\overline{\mathbf{G}}_b$, for fixed values of n and m, were calculated as a function of $h\tau/r$ for an isotropic material and as a function of τ for an anisotropic silicon crystal. For the isotropic material, the ratio of the speeds of the longitudinal wave (P-wave) and transverse wave (S-wave) was taken as $\sqrt{3}$. The elastic constants of silicon used for calculations were $C_{11} = 165$ GPa, $C_{12} = 63$ GPa and $C_{44} = 79$ GPa with respect to the symmetry axes. The coordinate axes were chosen such that the (x_1, x_3)-plane was on the $(1\,1\,1)$ surface and the x_1-axis was in the $[1\,\bar{1}\,0]$ direction. In all the figures presented sharp peaks correspond to singularities arising from the wave arrivals.

Figs. 2 and 3 show $(\overline{\mathbf{G}}_f)_{12}$ and $(\overline{\mathbf{G}}_f)_{22}$, respectively, for the isotropic material at $n = 1$ and $m = 0.5$. The components $(\overline{\mathbf{G}}_f)_{12}$ and $(\overline{\mathbf{G}}_f)_{22}$, respectively, correspond to the horizontal and vertical displacements due to a unit vertical force. The arrivals of direct P-wave and S-wave are indicated by P and S. The arrivals of the

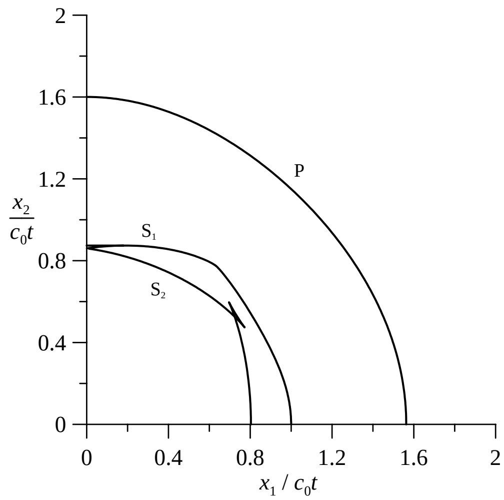

Fig. 4. The wave surface in an infinite medium of silicon.

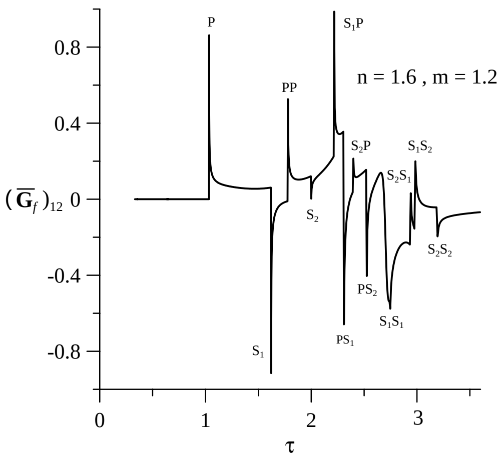

Fig. 5. $(\overline{\mathbf{G}}_f)_{12}$ for silicon at $n = 1.6$ and $m = 1.2$.

reflected P-wave and S-wave due to the incident P-wave are indicated, respectively, by PP and PS; those due to the incident S-wave are indicated by SP and SS. The result is in close agreement with that in (Su and Farris, 1994).

Fig. 4 displays the wave surface in an infinite medium of silicon. The wave surface consists of three wave fronts: one corresponding to quasi-longitudinal P-wave and two corresponding to quasi-shear S_1-wave and S_2-wave. Figs. 5 and 6 show $(\overline{\mathbf{G}}_f)_{12}$ and $(\overline{\mathbf{G}}_f)_{22}$, respectively, for silicon at $n = 1.6$ and $m = 1.2$. The components $(\overline{\mathbf{G}}_f)_{12}$ and $(\overline{\mathbf{G}}_f)_{22}$, respectively, are the horizontal displacement and vertical displacement due to a line impulse in the vertical direction. Figs. 7 and 8 show $(\overline{\mathbf{G}}_b)_{11}$ and $(\overline{\mathbf{G}}_b)_{21}$, respectively, for silicon at $n = 1.3$ and $m = 1.2$. The components $(\overline{\mathbf{G}}_b)_{11}$ and $(\overline{\mathbf{G}}_b)_{21}$, respectively, are the horizontal displacement and vertical displacement due to a line dislocation of a unit Burgers vector in the horizontal direction. In Figs. 5–8, the arrivals of three direct waves are indicated by P, S_1, and S_2. The arrivals of the reflected

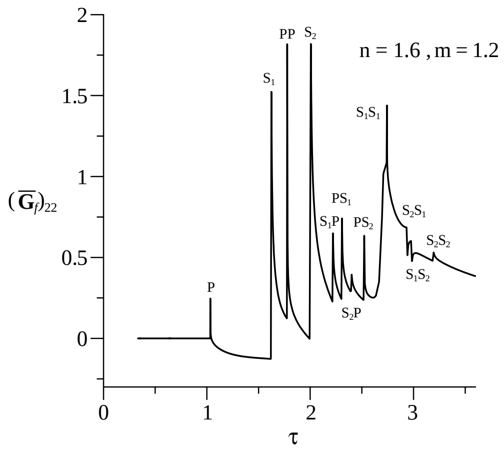

Fig. 6. $(\overline{\mathbf{G}}_f)_{22}$ for silicon at $n = 1.6$ and $m = 1.2$.

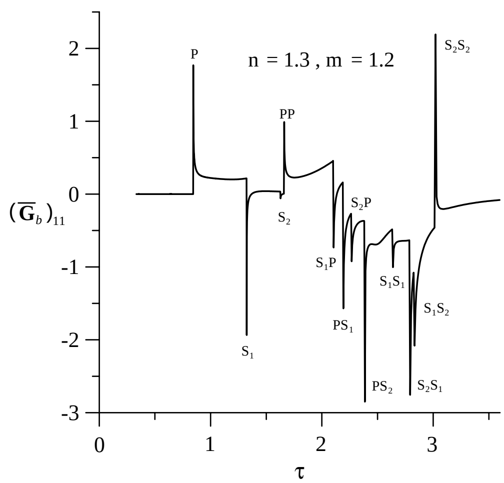

Fig. 7. $(\overline{\mathbf{G}}_b)_{11}$ for silicon at $n = 1.3$ and $m = 1.2$.

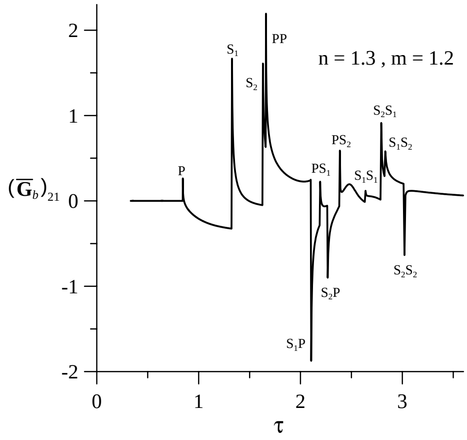

Fig. 8. $(\overline{\mathbf{G}}_b)_{21}$ for silicon at $n = 1.3$ and $m = 1.2$.

waves due to the incident P-wave are indicated by PP, PS_1, PS_2. The arrivals of the reflected waves due to the incident S_1 and S_2 waves are indicated in a similar way. It is seen that all wave arrivals are accurately captured.

5. Conclusion

The formulation developed by Wu (2000) is generalized to treat buried dynamic sources in an anisotropic elastic half-space. The displacement or traction fields in time domain have been obtained without using integral transforms. The numerical results show that the dynamic responses can be efficiently calculated and the complicated wave phenomena accurately captured.

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